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# Dimensional Continuation of Gauge-Invariant Quantities in Yang-Mills Theory*

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**Abstract**

Consideration of some perturbatively calculated gauge-invariant expectation values of local noncomposite operators in pure Yang-Mills theory indicates that those expectation values which are not dimension specific, and which are well defined near $D = 4$ dimensions, have some *finite* limit as the evaluated expression is formally extrapolated from $D = 4$ to $D = 2$. If this finite limit is a generic feature for such quantities it would not only be quite remarkable but also of some utility : For example, one may then use it as a convenient necessary condition when checking gauge invariance. Some examples are discussed at nonzero temperature.

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# 1 Introduction

Yang-Mills theory enables a consistent description of self-interacting spin one fields which form the backbone of both quantum chromodynamics and the electroweak theory. As is well known, the underlying fabric which holds Yang-Mills (YM) theory together is gauge-invariance, and in perturbative quantum field theory this invariance, or redundancy, implies that only the $(D-2)$ transverse components of the vector gauge field describe propagating degrees of freedom in $D$ dimensions. Indeed in any practical calculation one is forced to break the classical gauge-invariance by introducing a gauge-fixing term in the Lagrangian. However in the calculation of gauge-invariant quantities, the final result should be insensitive to the type of gauge-fixing and one often checks that this is indeed the case by using an arbitrary gauge-fixing parameter.

Amusingly, precisely because Yang-Mills theory is a gauge theory, it simplifies in a particular domain : that is for $D = 2$ dimensions. Then there are no physical particles and furthermore in the axial ($A_1 = 0$) gauge the $D = 2$ theory is manifestly noninteracting. A particular consequence of this is that gauge-invariant expectation values, of local operators, in $\mathrm{YM}_2$ are null since they are vanishing in the axial gauge. However nonlocal operators, such as the Wilson loop (or perhaps even appropriately constructed local but composite operators), can have nonzero gauge-invariant expectation values even in a free theory because such operators mimic external sources which couple to the gauge-fields. For example, the Coulombic potential energy between two external static charges in a theory with only photons is given by the large Euclidean time limit of the expectation value of the Wilson loop. In what follows, I will only consider local, noncomposite operators.

Suppose one could calculate some non-dimension-specific gauge-invariant quantity to all orders in perturbation theory for an arbitrary $D$ dimensions. Let's call this quantity $G(D)$. Then from the previous discussion it follows that $G(D \to 2) = 0$ and one expects this limit to be smooth. In practice, one can only compute $G^p(D_0; D)$ which is a perturbative approximation to $G(D)$ valid for some values of $D$ near $D = D_0$. The formal limit $G^p(D_0; D \to 2)$ then need not vanish simply because one is extrapolating beyond the region of validity of the calculation. For example, the two-loop free energy of a gluon plasma is easily calculated at and near $D = 4$ but the expression already diverges as $D \to 3^+$. Indeed, typically one finds $G^p(D_0 = 4; D \to 3) = \infty$, reminding one that the boundary of the region where the calculation is sensible has been approached and that the further formal extrapolation $D \to 2$ could, *a priori*, yield anything.

In Ref.[1], two infrared prescriptions were used in an attempt to *enforce* the limit $G^p(D_0 = 4; D \to 2) = 0$. The prescriptions appeared to be self-consistent in the examples considered but it was not clear if this would always be the case. Here I would

like to expand on an observation made in the conclusion of Ref.[1] : That without the *ad hoc* infrared prescriptions, the formal limit $G^p(D_0 = 4; D \to 2)$ was *finite* in the examples considered eventhough, as discussed in the preceeding paragraph, one might as well have expected a divergent result. Could it be that the examples were hinting at a general result ?. Perhaps it is true that perturbatively calculated gauge-invariant expectation values in pure YM theory which are not dimension specific, and which are well-defined near $D = 4$ dimensions, tend to a finite limit in the formal extrapolation $D \to 2$. For conciseness, the words of this question will be summarised as : "$G^p(D_0 = 4; D \to 2) \neq \infty$ ?". By the phrase 'well-defined' I exclude quantities which, for example, have collinear infrared singularities that are aggravated as $D \to 2$ [1].

In the next section some examples are enumerated with brief comments and in the concluding section I explain why the question "$G^p(D_0 = 4; D \to 2) \neq \infty$ ?" is 'nontrivial'. As the technical details and a more complete list of references for the examples below has already been given [1], I will attempt not to repeat them.

# 2 Examples

Here are some examples in $SU(N)$ YM theory at a nonzero temperature $T$. The quoted expressions represent sensible perturbative calculations for $3 < D \leq 4$ and are interpreted here by a formal extrapolation for $D < 3$. Note the singularity as $D \to 3^+$ which signals the breakdown of the naive perturbative calculation due to the appearance of infrared singularities.

*2.1 Free Energy*

The first three terms in the perturbative evaluation of the pressure (negative of the free energy) are

$$P = P_0 + P_2 + P_3 \,. \tag{2.1}$$

where

$$P_0 = (D-2)(N^2-1)\, T^D\, \pi^{-\frac{D}{2}}\, \Gamma(D/2)\, \zeta(D)\,, \tag{2.2}$$

$$P_2 = -\left(\frac{D-2}{2}\right)^2 g^2 N(N^2-1)\, T^{(2D-4)}\, \omega^2(D)\, I^2(D)\,, \tag{2.3}$$

$$P_3 = \frac{(N^2-1)T}{2}\Gamma\left(\frac{1-D}{2}\right)\left(\frac{m_{el}^2}{4\pi}\right)^{\frac{D-1}{2}}\,, \tag{2.4}$$

with the the functions $\omega(D)$ and $I(D)$ defined by

$$\omega(D) = \left[2^{(D-2)}\, \Gamma\left(\frac{D-1}{2}\right)\, \pi^{\frac{(D-1)}{2}}\right]^{-1}\,, \tag{2.5}$$

$$I(D) \;=\; \Gamma(D-2)\,\zeta(D-2). \tag{2.6}$$

The ideal gas result $P_0$ is actually valid for all $D \geq 2$ simply because it is an exact statement of an explicitly solvable model : YM theory at zero coupling. The explicit $(D-2)$ factors in $P_0$ and $P_2$ come from the Lorentz algebra while those in $P_3$ are only implicit in the definition of the electric mass $m_{el}$ (see below).

*2.2 Self Energy*

Consider the time-time component of the one-loop gluon self-energy in the static limit, $\Pi_{00}(k_0 = 0, \vec{k})$. In four dimensions, the first two terms in its low momentum expansion give the electric mass squared (at lowest order) and a linear term in $|\vec{k}|T$. Though both terms are gauge-fixing independent, let me add some clarification about the momentum dependent term : It has been shown to be the same in general covariant gauges, the strict Coulomb gauge [2, 3] and also in a class of static gauges [4]. In the background gauge its numerical value is different [2] but this is unsurprising because Greens functions in nonabelian gauge theories do not necessarily have the same physical (or gauge-invariant) interpretation in the presence of a background field. Next, the linear term is susceptible to the effects of resummation [3] because it is a subleading (relative to order $g^2$) quantity at low momentum. However if one focuses in the narrow window $m_{el} \ll |\vec{k}| \ll T$ then resummation is unnecessary. For $D$ near four, the two terms are

$$m_{el}^2 \;=\; (D-2)\; g^2\; N\; T^{(D-2)}\omega(D)\;\Gamma(D-1)\zeta(D-2)\,, \tag{2.7}$$

and

$$T\left(\frac{D-2}{4}\right)\frac{k^{(D-3)}}{(4\sqrt{\pi})^{(D-4)}}\frac{1}{\Gamma(D/2)\cos(\pi D/2)}\,. \tag{2.8}$$

The $(D-2)$ factor in the momentum dependent term does *not* come from the Lorentz algebra but rather from the integrals in dimensional reguralisation. The function $\omega(D)$ is the same as defined above for the free energy.

*2.3 Hard Thermal Loops*

The electric mass to lowest order is the static limit of a more general gauge-invariant quantity : the leading $T^2$, momentum dependent, piece of the one loop gluon self-energy. This "hard thermal loop" (HTL) in the two-point function is in turn just one of an infinite set of gauge-invariant $n-$point ($n \geq 2$) HTL's [5]. These HTL's may be summarised by a nonlocal generating functional, for which an expression in $D$ dimensions may be written [6]. The divergence of the expression in Ref.[6] at $D = 3$ is related to the fact that "soft thermal loops" are then no longer suppressed relative to the HTL's. (As an aside, the HTL in the photon self-energy in QED is well-defined for all $2 < D \leq 4$ because of Pauli repulsion).

## 3 Conclusion

The question of course is whether there are more examples to support the suggestion $G^p(D_0 = 4; D \to 2) \neq \infty$ , or if one has to impose more conditions in order to separate the "bad apples" from the "chosen ones". If indeed some simple general statement could be made along the lines indicated in the abstract, then the result could be used as a convenient check of gauge-invariance. This was the motivation for the study in Ref.[1].

It is noteworthy that a nondiverging limit is obtained for $G^p(D_0; D \to 2)$, at least for the examples considered so far. If one were to take a similar limit for gauge *non*-invariant quantities then very often the result is divergent : For example the $D \to 2$ limit of Eq.(3) but excluding the ghost contribution, or the same limit for the one-loop gluon self-energy (including ghosts) at zero temperature, both diverge.

Finally, for nonlocal operators such as the Wilson loop one can have $\infty > G(D = 2) > 0$. One might ask if still, $G^p(D_0; D \to 2) \neq \infty$ ?. For the heavy quark-antiquark potential at lowest order, the answer is yes [7].

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